TWMS J. Pure Appl. Math. V.11, N.1, 2020, pp.43-56

USING (JCLR)-PROPERTY TO PROVE HYBRID FIXED POINT THEOREMS VIA QUASI F-CONTRACTIONS

HEMANT KUMAR NASHINE[1,2], MOHAMMAD IMDAD[3], MD. AHMADULLAH[4]

Abstract. The purpose of this paper is to prove some coincidence and common fixed point results for two hybrid pairs of coincidentally idempotent and quasi-coincidentally commuting mappings satisfying multi-valued F-contraction condition using joint common limit range property. We also prove some results for hybrid pairs of mappings which satisfy an F-contractive condition of Hardy-Rogers type. Consequently, a host of existing results are generalized and improved. Furthermore, we adopt some examples to demonstrate the realized improvements in our results proved herein.

Keywords: metric space, multi-valued mappings, quasi-coincidentally commuting mappings, common limit range property, common fixed point.

AMS Subject Classification: 47H09, 47H10, 54H25.

1. Introduction

The important Banach contraction principle is one of the cornerstones in the development of Nonlinear Analysis. Metric fixed point theory continues to be an active area of research under the ambit of non-linear analysis. The Banach contraction principle remains a source of inspiration for the researchers of this domain which was established by Banach [7] in 1922. Therefore, generalizations of the Banach contraction principle have been explored heavily by many authors.

Von Neumann originally initiated the fixed point theory for multivalued mappimgs in the study of game theory. Fixed point results for multivalued mappings are quite useful in control theory and have been frequently used in solving many problems of economics and game theory.

The development of the geometric fixed point theory for multivalued mappings was initiated with the work of Nadler [29] in 1969. He used the concept of Hausdorff metric to establish the multivalued contraction principle containing the Banach contraction principle as a particular case, as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and a mapping T from X into $CB(X)$ such that for all $x, y \in X$,*

$$\mathcal{H}(Tx, Ty) \leq \lambda d(x, y),$$

where $\lambda \in [0, 1)$. Then T has a fixed point, that is, there exists a point $x \in X$ such that $x \in Tx$.

[1,2] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, India and Department of Pure and Applied Mathematics, University of Johannesburg, Kingsway Campus South Africa

[3]Department of Mathematics, Aligarh Muslim University, Aligarh, India

[4]Department of Mathematics, Aligarh Muslim University, Aligarh, India and Department of Mathematics Rammohan College, Kolkata, India

e-mail: drhknashine@gmail.com, hemant.nashine@vit.ac.in, mhimdad@gmail.com mhimdad@yahoo.com ahmadullah2201@gmail.com

Manuscript received October 2017.

The first ever use of a weak commutativity condition in a hybrid setting can be traced back to a paper [21] due to Itoh and Takahashi in 1977 while the regular use of a weak commutativity condition mostly belongs to Sessa [40] which appeared in 1982. Kaneko and Sessa [27] weakened the notion of weak commutativity by extending the idea of compatibility (due to Jungck [22]) to a hybrid pair of mappings. Pathak [34] extended the concept of compatibility (due to Jungck [23]) by defining weak compatibility for hybrid pairs of mappings (including single-valued case also) and utilized the same to prove results on the existence of coincidence and common fixed points. Several authors have proved coincidence and common fixed point theorems in metric spaces satisfying hybrid-type contraction conditions (e.g. [6, 9, 10, 11, 34, 35, 44]).

It is well known that strict contractive conditions do not ensure the existence of fixed points unless the underlying space is assumed to be compact relatively more substantial conditions replaced the contractive conditions. In 2004, Kamran [25] extended the idea of the property (E.A) (due to Aamri and Moutawakil [1]) to a hybrid pair of mappings and proved some fixed point results. Imdad and Ali [17] pointed out that the property (E.A) buys the suitable required containment between the range of one mapping into the range of other mappings of the pair. In 2005, Liu et al. [28] investigated a new property for two hybrid pairs of mappings and term the same as common property (E.A) which reduces to the property (E.A) whenever restricted to a single pair. By using this interesting property, they extended the results of Kamran [25]. Also, Ali and Imdad [5] studied the notion of non-compatible mappings (due to Pant [32]) in the hybrid setting.

In 2011, Samet and Vetro [39] pointed out an error in the proof of Theorem 1 of Rhoades et al. [38] and proved some results on coincidence points for a hybrid pair of mappings satisfying ϕ-contractive condition in the presence of the property (E.A). Damjanović et al. [8] obtained a coincidence point theorem for two hybrid pairs of mappings which improved the results of Gordji et al. [13]. Sintunavarat and Kumam [47] proposed the idea of 'common limit range property' for single-valued mappings which never demands the completeness (or closedness) of the underlying subspaces. Most recently, Imdad et al. [19] defined the notion of common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric spaces. Motivated by the idea of Liu et al. [28], Imdad et al. [20] extended the notion of common limit range property to pair of self mappings and obtained some fixed point theorems in Menger and metric spaces. In the recent past, several authors have contributed to the vigorous development of metric fixed point theory for hybrid pair of mappings (e.g. [4, 5, 12, 14-17, 26, 30, 36, 42, 43, 45, 46, 48, 49, 52]).

The paper aims to prove some coincidence and common fixed point theorems for two hybrid pairs of coincidentally idempotent and quasi-coincidentally commuting mappings satisfying joint common limit range property. We also prove some results for hybrid pairs of mappings which satisfy an F-contractive condition of Hardy-Rogers type. Consequently, a host of existing results are generalized and improved. Furthermore, we adopt some examples to demonstrate the realized improvements in our results proved herein.

2. Preliminaries

The following definitions and results are needed in the subsequent discussion.

Definition 2.1. *Let (X, d) be a metric space. A subset A of X is said to be*

(*i*) *closed if $A = \overline{A}$ where $\overline{A} = \{x \in X : d(x, A) = 0\}$,*

(*ii*) *bounded if $\delta(A) < \infty$ where $\delta(A) = \sup\{d(a, b) : a, b \in A\}$.*

Let (X, d) be a metric space. Then, on the lines of Nadler [29], we adopt

- 2^X : the collection of all subsets of X,
- $\mathcal{CL}(X) = \{A : A$ is a non-empty closed subset of $X\}$,
- $\mathcal{CB}(X) = \{A : A$ is a non-empty closed and bounded subset of $X\}$,
- for non-empty closed and bounded subsets A, B of X and $x \in X$,

$$d(x, A) = \inf\{d(x, a) : a \in A\},$$

and

$$\mathcal{H}(A, B) = \max\left\{\{\sup d(a, B) : a \in A\}, \{\sup d(A, b) : b \in B\}\right\}.$$

It can be pointed out that $\mathcal{CB}(X)$ is a metric space equipped with the distance $\mathcal{H}$ which is known as the Hausdorff-Pompeiu metric on $\mathcal{CB}(X)$ provided (X, d) is a metric space.

Definition 2.2. [32] *Let (X, d) be a metric space with $T : X \to \mathcal{CB}(X)$ and $g : X \to X$. The hybrid pair (T, g) is said to be an R-weakly commuting if for given $x \in X$, $gTx \in CB(X)$, there exists some positive real number R such that $\mathcal{H}(Tgx, gTx) \leq Rd(Tx, gx)$.*

Definition 2.3. [27] *Let (X, d) be a metric space with $T : X \to \mathcal{CB}(X)$ and $g : X \to X$. The hybrid pair (T, g) is said to be a compatible if $gTx \in \mathcal{CB}(X)$ for all $x \in X$ and $\lim_{n\to\infty} \mathcal{H}(Tgx_n, gTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X, for some $t \in X$ and $A \in \mathcal{CB}(X)$ such that $\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Tx_n$.*

Here it can be pointed out that compatible mappings need not be R-weakly commuting (see [32]). Also, on the points of coincidence R-weak commutativity is equivalent to commutativity and remains a necessary minimal condition for the existence of common fixed points for contractive type mappings.

Definition 2.4. [5] *Let (X, d) be a metric space with $T : X \to \mathcal{CB}(X)$ and $g : X \to X$. The hybrid pair (T, g) is said to be a non-compatible if there exists at least one sequence $\{x_n\}$ in X, for some $t \in X$ and $A \in \mathcal{CB}(X)$ such that $\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Tx_n$ but $\lim_{n\to\infty} \mathcal{H}(Tgx_n, gTx_n)$ is either non-zero or nonexistent.*

Now we define the following definitions for non-self mappings:

Definition 2.5. [24, 34] *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to 2^X$ and $g : Y \to X$. The hybrid pair (T, g) is said to be a weakly compatible if they commute at their coincidence points, that is, $gTx = Tgx$ whenever $gx \in Tx$.*

Definition 2.6. [15] *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to 2^X$ and $g : Y \to X$. The hybrid pair (T, g) is said to be a quasi-coincidentally commuting if $gx \in Tx$ (for $x \in X$ with $Tx, gx \in Y$) implies gTx is contained in Tgx.*

Definition 2.7. [15] *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to 2^X$ and $g : Y \to X$. The mapping g is said to be a coincidentally idempotent with respect to the mapping T, if $gx \in Tx$ with $gx \in Y$ imply $ggx = gx$, that is, g is idempotent at coincidence points of the pair (T, g).*

Definition 2.8. [25] *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to \mathcal{CB}(X)$ and $g : Y \to X$. Then the hybrid pair (T, g) is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in Y, for some $t \in X$ and $A \in \mathcal{CB}(X)$ such that*

$$\lim_{n\to\infty} gx_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Definition 2.9. [28] *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T, S : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$. Then the hybrid pairs (T, f) and (S, g) are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y, for some $t \in Y$ and $A, B \in \mathcal{CB}(X)$ such that*

$$\lim_{n\to\infty} Tx_n = A, \lim_{n\to\infty} Sy_n = B, \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B.$$

Definition 2.10. [18] *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to \mathcal{CB}(X)$ and $g : Y \to X$. Then the hybrid pair (T, g) is said to satisfy (CLRg) property if there exists a sequence $\{x_n\}$ in Y, for some $u \in Y$ and $A \in \mathcal{CB}(X)$ such that*

$$\lim_{n\to\infty} gx_n = gu \in A = \lim_{n\to\infty} Tx_n.$$

One may notice that the notion of common property (E.A) requires the closeness of the underlying subspaces to ascertain the existence of coincidence points. To remove this requirement, Imdad et al. [18] introduced the notion of Joint Common Limit Range Property (in short (JCLR) property) for two hybrid pairs of non-self mappings as follows:

Definition 2.11. [18] *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T, S : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$. Then the hybrid pairs (T, f) and (S, g) are said to have the (JCLR) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A, B \in \mathcal{CB}(X)$ such that*

$$\lim_{n\to\infty} Tx_n = A, \lim_{n\to\infty} Sy_n = B, \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A \cap B \cap f(Y) \cap g(Y),$$

i.e., there exist u and v in Y such that $t = fu = gv \in A \cap B$.

Inspired by Imdad et al. [18], we present some examples which demonstrate the utility of preceding definition.

Example 2.1. *Consider $Y = [0, 1] \subset [0, \infty) = X$ equipped with the usual metric. Define $F, G : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$ as follows:*

$$fx = \begin{cases} 1 - x, & \text{if } 0 \le x \le \frac{1}{2}; \\ \frac{4}{5}, & \text{if } \frac{1}{2} < x \le 1. \end{cases} \qquad gx = \begin{cases} 1 - x^2, & \text{if } 0 \le x < \frac{1}{2}; \\ \frac{1}{2}, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

$$Fx = \begin{cases} \left[\frac{1}{2}, \frac{3}{4}\right], & \text{if } 0 \le x \le \frac{1}{2}; \\ \left[\frac{1}{4}, \frac{1}{2}\right], & \text{if } \frac{1}{2} < x \le 1. \end{cases} \qquad Gx = \begin{cases} \left[\frac{1}{2}, \frac{3}{5}\right], & \text{if } 0 \le x < \frac{1}{2}; \\ \left[\frac{2}{5}, \frac{x+1}{2}\right], & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

If we choose the esteemed sequences $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $\{y_n\} = \left\{\frac{1}{2} + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ in Y, then one can verify that the pairs (F, f) and (G, g) enjoy the (JCLR) property i.e.,

$$\lim_{n\to\infty} Fx_n = \left[\frac{1}{2}, \frac{3}{4}\right], \lim_{n\to\infty} Gy_n = \left[\frac{2}{5}, \frac{3}{4}\right], \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \frac{1}{2},$$

where $f\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = \frac{1}{2} \in \left[\frac{1}{2}, \frac{3}{4}\right]$.

Notice that the (JCLR) property implies the common property (E.A), but the converse implication is not vaild in general. The following example substantiates this view point.

Example 2.2. *In the setting of Example 2.1, replace the mappings f and g (besides retaining the rest):*

$$fx = \begin{cases} 1-x, & \text{if } 0 \le x < \frac{1}{2}; \\ \frac{4}{5}, & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \quad gx = \begin{cases} 1-x^2, & \text{if } 0 \le x \le \frac{1}{2}; \\ \frac{1}{2}, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

If we consider the sequences as in Example 2.1, then one can verify that

$$\lim_{n\to\infty} Fx_n = \left[\frac{1}{2}, \frac{3}{4}\right], \lim_{n\to\infty} Gy_n = \left[\frac{2}{5}, \frac{3}{4}\right], \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \frac{1}{2}(= t),$$

where $\frac{1}{2} \in \left[\frac{1}{2}, \frac{3}{4}\right]$. *Hence both the pairs* (F, f) *and* (G, g) *share the common property (E.A). However, the pairs* (T, f) *and* (S, g) *do not satisfy (JCLR) property due to the fact that there does not exists any point* u *in* Y *such that* $t = fu$.

For the sake of completeness, we use the following notions.

Throughout the article, respectively $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ stand for the set of all real numbers, the set of all positive real numbers and the set of all positive integers. In what follows, $\mathcal{F}$ stands for the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions

- (F_1) F is strictly increasing;
- (F_2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
- (F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Some examples of such functions $F \in \mathcal{F}$ are $F(t) = \ln t$, $F(t) = t + \ln t$, $F(t) = -1/\sqrt{t}$, see [51].

In [37], Piri and Kumam replaced the condition (F_3) with continuity of F.

We denote by $\mathfrak{F}$, the family of all strictly increasing and continuous functions $F : \mathbb{R}^+ \to \mathbb{R}$. Here, it can be pointed out that the family $\mathfrak{F}$ is different from the family $\mathcal{F}$.

Definition 2.12. [51] *Let* (X, d) *be a metric space. A self-mapping* T *on* X *is called an* F*-contraction if there exist* $F \in \mathcal{F}$ *and* $\tau \in \mathbb{R}^+$ *such that*

$$\tau + F(d(Tx, Ty)) \le F(d(x, y)), \tag{1}$$

for all $x, y \in X$ *with* $d(Tx, Ty) > 0$.

Definition 2.13. [41] *Let* (X, d) *be a metric space. A multivalued mapping* $T : X \to \mathcal{CL}(X)$ *is called a generalized* F*-contraction if there exist* $F \in \mathcal{F}$ *and* $\tau \in \mathbb{R}^+$ *such that (for all* $x, y \in X$ *with* $y \in Tx$*, for some* $z \in Ty$ *with* $d(y, z) > 0$*)*

$$\tau + F(d(y, z)) \le F\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}\right). \tag{2}$$

Observe that, on choosing $F(x) = \ln x$ on Definition 2.13, the condition (2) reduces to the following (for all $x, y \in X$, $z \in Ty$, $y \ne z$)

$$d(y, z) \le e^{-\tau} \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

It is clear that for $z, y \in X$ such that $y = z$ the previous inequality also holds.

Some fixed point results for single-valued (resp. multivalued) F-contraction are obtained in [3, 51, 37] (resp. [2, 41, 31, 50]).

3. Main results

The attempted improvements in this paper are four-fold:

- quasi-coincidentally commuting and coincidentally idempotent notion is used, which is weaker than coincidentally idempotent in the case when the set of coincidence points is not empty;
- joint common limit range property (JCLR) is utilized (instead of common property (E.A));
- any requirement of closedness of the range of f is relaxed;
- a new kind of contractive conditions (so-called F-contraction conditions) are used, that originated in the work of Wardowski [51].

This section is divided into two parts. In the first subsection, we prove a common fixed point theorem for two hybrid pairs of quasi-coincidentally commuting and coincidentally idempotent mappings satisfying multi-valued generalized F-contractions condition via joint common limit range property in metric spaces. In contrast, in the second subsection, we obtain some results for two hybrid pairs of mappings which satisfy F-contractive condition of Hardy-Rogers-type.

3.1. **Result - I.** Our first main result is as follows:

Theorem 3.1. *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T, S : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$. Assume that there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}(Tx, Sy)) \leq F\left(\max\left\{d(fx, gy), d(fx, Tx), d(gy, Sy), d(fx, Sy), d(gy, Tx)\right\}\right), \qquad (3)$$

for all $x, y \in X$ with $\mathcal{H}(Tx, Sy) > 0$. Suppose that the pairs (T, f) and (S, g) enjoy the (JCLR) property. Then the each pairs (T, f) and (S, g) have a coincidence point .

Moreover, if $Y \subset X$ and the pairs (T, f) and (S, g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (T, f) and (S, g) have a common fixed point in X.

Proof. Since the hybrid pairs (T, f) and (S, g) enjoy the (JCLR) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A, B \in \mathcal{CB}(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n, \lim_{n\to\infty} gy_n = t \in B = \lim_{n\to\infty} Sy_n,$$

for some $u, v \in Y$ and $t = fv = gu \in A \cap B$. We assert that $gu \in Su$. If not, then using condition (3), we get

$$\tau + F(\mathcal{H}(Tx_n, Su)) \leq F\left(\max\left\{d(fx_n, gu), d(fx_n, Tx_n), d(gu, Su), d(fx_n, Su), d(gu, Tx_n)\right\}\right).$$

Taking the limit as $n \to \infty$, we have

$$\tau + F(\mathcal{H}(A, Su)) \leq F\left(\max\left\{0, d(gu, A), d(gu, Su), d(gu, Su), d(gu, A)\right\}\right).$$

Since $\tau > 0$ and F is strictly increasing, we have

$$\mathcal{H}(A, Su) < \max\left\{0, 0, d(gu, Su), d(gu, Su), 0\right\} = d(gu, Su).$$

Since $t = fv = gu \in A \cap B$, it follows from the definition of Hausdorff metric that

$$d(gu, Su) \leq \mathcal{H}(A, Su) < d(gu, Su),$$

a contradiction. Hence $gu \in Su$ which shows that the pair (S, g) has a coincidence point u in Y.

Now we show that $fv \in Tv$, if not, then using inequality (3), one obtains

$$\begin{aligned} F(d(Tv, fv)) &= F(d(Tv, gu)) \\ &< \tau + F(\mathcal{H}(Tv, Su)) \\ &\leq F\left(\max\left\{d(fv, gu), d(fv, Tv), d(gu, Su), d(fv, Su), d(gu, Tv)\right\}\right) \\ &= F\left(\max\left\{0, d(fv, Tv), 0, 0, d(fv, Tv)\right\}\right) \\ &= F(d(Tv, fu)), \end{aligned}$$

a contradiction. Hence $fv \in Tv$ which shows that the pair (T, f) has a coincidence point v in Y.

Suppose that $Y \subset X$. Since v is a coincidence point of the pair (T, f), which is quasi-coincidentally commuting and coincidentally idempotent with respect to mapping T, we have $fv \in Tv$ and $ffv = fv$, therefore $fv = ffv \in f(Tv) \subset T(fv)$ which shows that fv is a common fixed point of the pair (T, f). Similarly, u is a coincidence point of the pair (S, g) which is quasi-coincidentally commuting and coincidentally idempotent concerning mapping S, one can easily show that gu is a common fixed point of the pair (S, g). The analogous arguments work for the alternate statement as well. This completes the proof. □

Now, we obtain the following corollaries.

Corollary 3.1. *Let (X, d) be a metric space whereas Y an arbitrary non-empty set. Let $T, S : Y \to \mathcal{CB}(X)$ be upper semicontinuous and $f, g : Y \to X$. Assume that there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}(Tx, Sy)) \leq F\left(\max\left\{d(fx, gy), d(fx, Tx), d(gy, Sy), d(fx, Sy), d(gy, Tx)\right\}\right), \qquad (4)$$

for all $x, y \in X$ with $\mathcal{H}(Tx, Sy) > 0$. Suppose that the pairs (T, f) and (S, g) enjoy the (JCLR) property. Then the each hybrid pairs (T, f) and (S, g) have a coincidence point.

Moreover, if $Y \subset X$ and the pairs (T, f) and (S, g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (T, f) and (S, g) have a common fixed point in X.

Proof. All the requirements of Theorem 3.1 are fulfilled and hence the result follows on the same line of proof of Theorem 3.1. □

Corollary 3.2. *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T, S : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$. Suppose that the hybrid pairs (T, f) and (S, g) share the common property (E.A) and satisfy inequality (3). If $f(Y)$ and $g(Y)$ are closed subsets of X, then the each pairs (T, f) and (S, g) have a point of coincidence.*

In particular, if $Y \subset X$ and the pairs (T, f) and (S, g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (T, f) and (S, g) have a common fixed point in X.

Proof. If the pairs (T, f) and (S, g) share the common property (E.A), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and some $t \in X$, $A, B \in \mathcal{CB}(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n, \lim_{n\to\infty} gy_n = t \in B = \lim_{n\to\infty} Sy_n.$$

As $f(Y)$ and $g(Y)$ are closed subsets of X, there exist u and v in X such that $t = fu = gv$ for some $u, v \in Y$. Hence the hybrids pairs (T, f) and (S, g) satisfy the (JCLR) property. The rest of the proof runs on the lines of the proof of Theorem 3.1. □

Notice that a non-compatible hybrid pair always satisfies the property (E.A). Hence, we get the following corollary.

Corollary 3.3. *Let (X,d) be a metric space whereas Y an arbitrary non-empty set with $T,S : Y \to \mathcal{CB}(X)$ and $f,g : Y \to X$. Suppose that the hybrid pairs (T,f) and (S,g) share the non-compatible property and satisfy inequality (3). If $f(Y)$ and $g(Y)$ are closed subsets of X, then the each pairs (T,f) and (S,g) have a point of coincidence.*

In particular, if $Y \subset X$ and the pairs (T,f) and (S,g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (T,f) and (S,g) have a common fixed point in X.

3.2. **Result - II.** Our second main result is as follows:

Theorem 3.2. *Let (X,d) be a metric space whereas Y an arbitrary non-empty set with $T,S : Y \to \mathcal{CB}(X)$ and $f,g : Y \to X$. Assume that there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}(Tx,Sy)) \leq F(\alpha\, d(fx,gy) + \beta\, d(fx,Tx) + \gamma\, d(gy,Sy) + \delta\, d(fx,Sy) + \varepsilon\, d(gy,Tx)), \quad (5)$$

for all $x,y \in X$ with $\mathcal{H}(Tx,Sy) > 0$ and $\alpha,\beta,\gamma,\delta,\varepsilon \geq 0$ with $\alpha+\beta+\gamma+\delta+\varepsilon < 1$. Suppose that the pairs (T,f) and (S,g) enjoy the (JCLR) property. Then the each pairs (T,f) and (S,g) have a coincidence point.

Moreover, if $Y \subset X$ and the pairs (T,f) and (S,g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (T,f) and (S,g) have a common fixed point in X.

Proof. Since the pairs (T,f) and (S,g) enjoy the (JCLR) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A,B \in \mathcal{CB}(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n, \lim_{n\to\infty} gy_n = t \in B = \lim_{n\to\infty} Sy_n,$$

for some $u,v \in Y$ and $t = fv = gu \in A \cap B$. We assert that $gu \in Su$. If not, then using condition (5), one obtains

$$\begin{aligned} \tau + F(\mathcal{H}(Tx_n,Su)) &\leq F(\alpha\, d(fx_n,gu) + \beta\, d(fx_n,Tx_n) + \gamma\, d(gu,Su) \\ &\quad +\delta\, d(fx_n,Su) + \varepsilon\, d(gu,Tx_n)). \end{aligned} \quad (6)$$

Taking the limit as $n \to \infty$ in (6), we have

$$\tau + F(\mathcal{H}(A,Su)) \leq F((\gamma+\delta)d(gu,Su)).$$

Since $\tau > 0$ and F is strictly increasing, it follows that

$$d(gu,Su) \leq \mathcal{H}(A,Su) < (\gamma+\delta)d(gu,Su).$$

which is a contradiction as $\gamma+\delta < 1$. Hence $gu \in Su$ which shows that $u \in Y$ is a coincidence point of the pair (S,g).

Now we assert that $fv \in Tv$. On using inequality (5), one gets

$$\begin{aligned} F(d(Tv,fv)) &= F(d(Tv,gu)) \\ &< \tau + F(\mathcal{H}(Tv,Su)) \\ &\leq F(\alpha\, d(fv,gu) + \beta\, d(fv,Tv) + \gamma\, d(gu,Su) + \delta\, d(fv,Su) + \varepsilon\, d(gu,Tv)) \\ &= F((\beta+\varepsilon)\, d(fv,Tv)), \end{aligned}$$

which is a contradiction (as $\beta+\varepsilon < 1$). Then we have $fv \in Tv$ which shows that $v \in Y$ is a coincidence point of the pair (T,f).

The rest of the proof can be completed on the lines of the proof of Theorem 3.1. This completes the proof. □

Remark 3.1. *Choosing $\alpha,\beta,\gamma,\delta,\varepsilon$ suitably in (5) of Theorem 3.2, one can deduce a multitude of corollaries.*

In [33], Pant and Pant introduced the notion of conditionally commuting for a hybrid pair of mappings which is the weakest form of the commutativity. We unitize this concept to derive a new result. For this first, we define it.

Definition 3.1. [33] *Let (X,d) be a metric space whereas Y an arbitrary non-empty set with $F:Y\to 2^X$ and $f:Y\to X$. The hybrid pair (F,f) is said to be a conditionally commuting if they commute on a non-empty subset of the set of coincidence points whenever the set of their coincidences is non-empty.*

Theorem 3.3. *Let (X,d) be a metric space whereas Y an arbitrary non-empty set with $T,S: Y\to \mathcal{CB}(X)$ and $f,g:Y\to X$ satisfying inequality (5) where $F\in\mathfrak{F}$ and $\tau\in\mathbb{R}^+$. Suppose that the hybrid pairs (T,f) and (S,g) enjoy the (JCLR) property. Then the each pairs (T,f) and (S,g) have a point of coincidence.*

Moreover, if $Y\subset X$, then the pairs (T,f) and (S,g) have a common fixed point provided the pairs (T,f) and (S,g) are conditionally commuting.

Proof. In view of proof of Theorem 3.2, the each pairs (T,f) and (S,g) have a coincidence point u,v in Y. Suppose that $Y\subset X$. Since the pair (T,f) is conditionally commuting, two possible cases arise:

Case I: The pair (T,f) commutes at $v\in Y\subset X$, then $fv\in Tv$ so that $ffv\in f(Tv)\subset T(fv)$. Now we show that fv is a common fixed point of the pair (T,f). If it is not so, then using inequality (5), one gets

$$\begin{aligned}\tau+F(\mathcal{H}(Tfv,Sy_n)) &\leq F(\alpha\, d(ffv,gy_n)+\beta\, d(ffv,Tfv)+\gamma\, d(gy_n,Sy_n)\\ &\quad +\delta\, d(ffv,Sy_n)+\varepsilon\, d(gy_n,Tfv)).\end{aligned}$$

Taking the limit as $n\to\infty$, we have

$$\begin{aligned}\tau+F(\mathcal{H}(Tfv,B)) &\leq F(\alpha\, d(ffv,fv)+\beta\, d(ffv,Tfv)+\gamma\, d(fv,B)\\ &\quad +\delta\, d(ffv,B)+\varepsilon\, d(fv,Tfv))\\ &= F(\alpha\, d(ffv,fv)+\delta\, d(ffv,B)+\varepsilon\, d(fv,Tfv)).\end{aligned}$$

Since $\tau>0$ and F is strictly increasing, it follows that

$$\mathcal{H}(Tfv,B)) \leq (\alpha+\delta+\varepsilon)\, d(ffv,fv).$$

Since $t=fv=gu\in A\cap B$ and $ffv\in Tfv$, it follows (owing to the definition of Hausdorff metric) that

$$d(ffv,fv)\leq \mathcal{H}(Tfv,B)\leq(\alpha+\delta+\varepsilon)d(ffv,fv),$$

a contradiction (as $\alpha+\delta+\varepsilon<1$). Hence $fv=ffv\in Tfv$ which shows that fv is a common fixed point of the pair (T,f).

Case II: If T and f do not commute at v, then by virtue of conditional commutativity of T and f, there exists a coincidence point of T and f at which T and f commute, *i.e.*, there exists a point $v^{'}$ in Y such that $fv^{'}\in Tv^{'}$ and $ffv^{'}\in f(Tv^{'})\subset T(fv^{'})$. Rest of the proof can be completed on the lines of the Case I when T and f commute at v.

Similarly, we can show that gu is a common fixed point of the pair (S,g). This completes the proof of the theorem. □

On the setting $f=g$, in (3) of Theorem 3.1 we deduces the following corollary:

Corollary 3.4. *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T, S : Y \to \mathcal{CB}(X)$ and $g : Y \to X$. Assume that there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}(Tx, Sy)) \leq F\left(\max\left\{d(gx, gy), d(gx, Tx), d(gy, Sy), d(gx, Sy), d(gy, Tx)\right\}\right), \quad (7)$$

for all $x, y \in X$ with $\mathcal{H}(Tx, Sy) > 0$. Suppose that the pairs (T, g) and (S, g) enjoy the (JCLR) property. Then the each pairs (T, g) and (S, g) have a coincidence point .

Moreover, if $Y \subset X$ and the pairs (T, g) and (S, g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (T, g) and (S, g) have a common fixed point in X.

On the setting $T = S$, in (3) of Theorem 3.1 we deduces the following corollary:

Corollary 3.5. *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$. Assume that there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}(Tx, Ty)) \leq F\left(\max\left\{d(fx, gy), d(fx, Tx), d(gy, Ty), d(fx, Ty), d(gy, Tx)\right\}\right), \quad (8)$$

for all $x, y \in X$ with $\mathcal{H}(Tx, Ty) > 0$. Suppose that the pairs (T, f) and (T, g) enjoy the (JCLR) property. Then the each pairs (T, f) and (T, g) have a coincidence point .

Moreover, if $Y \subset X$ and the pairs (T, f) and (T, g) are quasi-coincidentally commuting and coincidentally idempotent, then the pairs (T, f) and (T, g) have a common fixed point in X.

On the setting $T = S$ and $f = g$, in (3) of Theorem 3.1 we deduces the following corollary:

Corollary 3.6. *Let (X, d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to \mathcal{CB}(X)$ and $f : Y \to X$. Assume that there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}(Tx, Ty)) \leq F\left(\max\left\{d(gx, gy), d(gx, Tx), d(gy, Ty), d(gx, Ty), d(gy, Tx)\right\}\right), \quad (9)$$

for all $x, y \in X$ with $\mathcal{H}(Tx, Ty) > 0$. Suppose that the pair (T, g) enjoys the (CLR_g) property. Then the pair (T, g) has a coincidence point .

Moreover, if $Y \subset X$ and the pair (T, g) is quasi-coincidentally commuting and coincidentally idempotent, then the pair (T, g) has a common fixed point in X.

4. Illustrative examples

Now we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results over some recently established results.

The following example exhibits the validity of conditions of Theorem 3.1. This example is inspired by Imdad et al. [18].

Example 4.1. *Let $Y = [0, 1] \subset [0, \infty) = X$ with the usual metric. Define $T, S : Y \to \mathcal{CB}(X)$ and $f, g : Y \to X$ as follows.*

$$Tx = \begin{cases} \left[\frac{1}{3}, \frac{3}{4}\right], & \text{if } 0 \leq x \leq \frac{1}{2}; \\ \left[\frac{1}{4}, \frac{1}{3}\right], & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \qquad Sx = \begin{cases} \left(\frac{1}{2}, \frac{3}{5}\right], & \text{if } 0 \leq x < \frac{1}{2}; \\ \left[\frac{2}{5}, \frac{1+x}{2}\right], & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

$$fx = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2}; \\ \frac{2x}{3}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \qquad gx = \begin{cases} 1 - \frac{x}{2}, & \text{if } 0 \leq x < \frac{1}{2}; \\ \frac{1}{2}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Choosing two sequences $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n\in\mathbb{N}}$ *and* $\{y_n\} = \left\{\frac{1}{2} + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ *in* Y*, one can see that the pairs* (T, f) *and* (S, g) *enjoy the (JCLR) property, i.e.*

$$\lim_{n\to\infty} f\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} \in \left[\frac{1}{3}, \frac{3}{4}\right] = \lim_{n\to\infty} T\left(\frac{1}{2} - \frac{1}{n}\right),$$
$$\lim_{n\to\infty} g\left(\frac{1}{2} + \frac{1}{n}\right) = \frac{1}{2} \in \left[\frac{2}{5}, \frac{3}{4}\right] = \lim_{n\to\infty} S\left(\frac{1}{2} + \frac{1}{n}\right),$$

where $\frac{1}{2} = f(\frac{1}{2}) = g(\frac{1}{2}) \in \left[\frac{2}{5}, \frac{3}{4}\right] = \left[\frac{1}{3}, \frac{3}{4}\right] \cap \left[\frac{2}{5}, \frac{3}{4}\right]$*. Now define* $F : \mathbb{R}^+ \to \mathbb{R}$ *be given by* $F(x) = \ln x$*. It is clear that* F *is* F *is strictly increasing and continuous. Then contractive condition* (3) *reduces to*

$$\mathcal{H}(Tx, Sy)) \leq e^{-\tau} \left(\max\left\{d(fx, gy), d(fx, Tx), d(gy, Sy), d(fx, Sy), d(gy, Tx)\right\}\right), \tag{10}$$

for all $x, y \in X$ *with* $\mathcal{H}(Tx, Sy) > 0$*. By a routine calculation one can show that the contractive condition* (10) *holds for every* $x \neq y \in X$ *and for some fixed* $\tau \in \mathbb{R}^+$*. Also it is cear that* $f(Y)$ *and* $g(Y)$ *are not closed subsets of* X*. The pairs* (T, f) *and* (S, g) *are quasi-coincidentally commuting at* $x = \frac{1}{2}$*, i.e.,* $f(\frac{1}{2}) \in T(\frac{1}{2})$*,* $fT(\frac{1}{2}) = \left(\frac{1}{3}, \frac{1}{2}\right] \subset \left[\frac{1}{3}, \frac{3}{4}\right] = Tf(\frac{1}{2})$ *and* $g(\frac{1}{2}) \in S(\frac{1}{2})$*,* $gS(\frac{1}{2}) = \left[\frac{3}{5}, \frac{3}{4}\right) \cup \left\{\frac{1}{2}\right\} \subset \left[\frac{2}{5}, \frac{3}{4}\right] = Sg(\frac{1}{2})$*. Thus, all conditions of Theorem 3.1 are satisfied and* $\frac{1}{2} = f(\frac{1}{2}) = g(\frac{1}{2}) \in T(\frac{1}{2}) = S(\frac{1}{2})$*.*

In the following illustration the importance of (JCLR) property for validity of the result is shown.

Example 4.2. *In the setting of Example 4.1, replace the mapping* f *by the following:*

$$fx = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x < \frac{1}{2}; \\ \frac{2x}{3}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then the pairs (T, f) *and* (S, g) *don't share the (JCLR) property as* $\frac{1}{3} = f(\frac{1}{2}) \neq \frac{1}{2} = g(\frac{1}{2}) \in \left[\frac{2}{5}, \frac{3}{4}\right] = \left[\frac{1}{3}, \frac{3}{4}\right] \cap \left[\frac{2}{5}, \frac{3}{4}\right]$*. And so there is no coincidence point of the pairs* (T, f) *and* (S, g)*.*

Now we furnish an example demonstrating that conditions of Theorem 3.1 is only sufficient and not necessary.

Example 4.3. *In the setting of Example 4.1, replace the mapping* S *by the following:*

$$Sx = \begin{cases} \left[\frac{1}{2}, \frac{3}{5}\right], & \text{if } 0 \leq x < \frac{1}{2}; \\ \left[\frac{2}{5}, \frac{1+x}{2}\right], & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then the pairs (T, f) *and* (S, g) *enjoy the (JCLR) property for two sequences* $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n\in\mathbb{N}}$ *and* $\{y_n\} = \left\{\frac{1}{2} + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ *in* Y*. Also* $f(Y)$ *and* $g(Y)$ *are not closed subsets of* X*. The pair* (T, f) *is quasi-coincidentally commuting at* $x = \frac{1}{2}$ *but* (S, g) *does not commute at* $x = \frac{1}{2}$ *as* $g(\frac{1}{2}) \in S(\frac{1}{2})$*,* $gS(\frac{1}{2}) = \left[\frac{5}{8}, \frac{3}{4}\right) \cup \left\{\frac{1}{2}\right\} \not\subseteq \left[\frac{2}{5}, \frac{3}{4}\right] = Sg(\frac{1}{2})$*. However, these four mappings have a coincidence at* $x = \frac{1}{2}$*, which also remains their common fixed point. This confirms that conditions of Theorem 3.1 is sufficient and not necessary.*

5. Conclusions

In this present study, some coincidence and common fixed point results are established for two hybrid pairs of mappings satisfying multi-valued F-contraction condition using joint common limit range property. After that some results are proved for hybrid pairs of mappings which satisfy an F-contractive condition of Hardy-Rogers type. Furthermore, we adopt some examples to demonstrate the realized improvements in our results proved herein. Consequently, a host of existing results are generalized and improved.

6. Acknowledgements

All the authors would like to extend their gratitude to both the anonymous referees for their valuable comments/suggestions.

References

[1] Aamri, M., El Moutawakil, D., (2002), Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., 270(1), pp.181-188.

[2] Acar, O., Durmaz, G., Minak, G., (2014), Generalized multivalued F-contractions on complete metric spaces, Bull. Iranian Math. Soc., 40(6), pp.1469-1478

[3] Ahmad, J., Al-Rawashdeh, A., Azam, A., (2015), New fixed point theorems for generalized F-contractions in complete metric spaces, Fixed Point Theory Appl., 2015:80, 18p.

[4] Ahmed, M.A., (2010), Common fixed points of hybrid maps and an application, Comput. Math. Appl., 60, pp.1888-1894.

[5] Ali, J., Imdad, M., (2010), Common fixed points of nonlinear hybrid mappings under strict contractions in semi-metric spaces, Nonlinear Anal. Hybrid Syst., 4(4), pp.830-837.

[6] Bae, J. S., (2003), Fixed point theorems for weakly contractive multivalued maps, J. Math. Anal. Appl., 284(2), pp.690-697.

[7] Banach, S., (1922), Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., 3, pp.133-181.

[8] Damjanović, B., Samet B., Vetro, C., (2012), Common fixed point theorems for multi-valued maps, Acta Math. Scientia, 32B(2), pp.818-824.

[9] Espínola R., Kirk, W. A., (2003), Set-valued contractions and fixed points, Nonlinear Anal., 54(3), pp.485-494.

[10] Fisher, B., (1981), Common fixed points of mappings and set-valued mappings, Rostock Math. Kolloq., 18, pp.69-77.

[11] Fisher, B., (1991), Common fixed points of commuting set-valued mappings, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 21(1), pp.147-156.

[12] George, R., Khan, M.S., Varghese, A., (2006), Fixed point theorems for hybrid mappings in metric spaces, Math. Morav., 10, pp.67-76.

[13] Gordji, M.E., Baghani, H., Khodaei, H., Ramezani, M., (2010), A generalization of Nadler's fixed point theorem, J. Nonlinear Sci. Appl., 3(2), pp.148-151.

[14] Imdad, M., Ahmed, M.A., (2012), Some common fixed point theorems for hybrid pairs of maps without the completeness assumption, Math. Slovaca, 62(2), pp.301-314.

[15] Imdad, M., Ahmad, A., Kumar, S., (2001), On nonlinear nonself hybrid contractions, Rad. Mat., 10(2), pp.233–244.

[16] Imdad, M., Ali, J., (2007), A general fixed point theorem for hybrid contractions via implicit functions, Southeast Asian Bull. Math., 31(1), pp.73-80.

[17] Imdad, M., Ali, J., (2008), Jungck's Common Fixed Point Theorem and E.A Property, Acta Math. Sinica (Engl. Ser.) 24(1), pp.87-94.

[18] Imdad, M., Chauhan, S., Kumam, P., (2015), Fixed point theorems for two hybrid pairs of non-self mappings under joint common limit range property in metric spaces, J. Nonlinear Convex Anal., 16(2), pp.243-254.

[19] Imdad, M., Chauhan, S., Soliman, A.H., Ahmed, M.A., (2014), Hybrid fixed point theorems in symmetric spaces via common limit range property, Demonstr. Math., 47(4), pp.949-962.

[20] Imdad, M., Pant, B.D., Chauhan, S., (2012), Fixed point theorems in Menger spaces using the (CLR_{ST}) property and applications, J. Nonlinear Anal. Optim., 3(2), pp. 225–237.

[21] Itoh, S., Takahashi, W., (1977), Single-valued mappings, multivalued mappings and fixed-point theorems, J. Math. Anal. Appl., 59(3), pp.514-521.

[22] Jungck, G., (1986), Compatible mappings and common fixed points, Internat. J. Math. Math. Sci., 9(4) pp.771-779.

[23] Jungck, G., (1996), Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East J. Math. Sci., 4(2), pp.199-215.

[24] Jungck, G., Rhoades, B.E., (1998), Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math., 29(3), pp.227-238.

[25] Kamran, T., (2004), Coincidence and fixed points for hybrid strict contractions, J. Math. Anal. Appl., 299(1), pp.235-241.

[26] Kamran, T., Cakić, N., (2008), Hybrid tangential property and coincidence point theorems, Fixed Point Theory, 9(2), pp.487–496.

[27] Kaneko, H., Sessa, S., (1989), Fixed point theorems for compatible multi-valued and single-valued mappings, Internat. J. Math. Math. Sci., 12(2), pp.257-262.

[28] Liu, Y., Wu J., Li, Z., (2005), Common fixed points of single-valued and multivalued maps, Internat. J. Math. Math. Sci., 19, pp.3045-3055.

[29] Nadler, S.B.Jr. (1969), Multivalued contraction mappings, Pacific J. Math., 20(2), pp.457-488.

[30] Naidu, S.V.R., (2003), Fixed point theorems for a broad class of multimaps, Nonlinear Anal., 52(3), pp.961-969.

[31] Nashine, H.K., Imdad, M., Ahmadullah, M., (2018), Common fixed point theorems for hybrid generalized (F, φ)-contractions under common limit range property with applications, Ukrainian Math. J., 69(11), pp.1784-1804 (Ukrainian Original Vol. 69, No. 11, 2017).

[32] Pant, R.P., (1994), Common fixed points of non-commuting mappings, J. Math. Anal. Appl., 188(2), pp.436-440.

[33] Pant, V., Pant, R.P., (2010), Common fixed points of conditionally commuting maps, Fixed Point Theory, 11(1), pp.113-118.

[34] Pathak, H.K., (1995), Fixed point theorems for weak compatible multi-valued and single-valued mappings, Acta Math. Hungar., 67(1-2), pp.69-78.

[35] Pathak, H.K., Kang, S.M., Cho, Y. J., (1998), Coincidence and Fixed point theorems for non-linear hybrid generalized contractions, Czechoslovak Math. J., 48, pp.341-357.

[36] Pathak, H.K., Khan, M.S., (2002), Fixed and coincidence points of hybrid mappings, Arch. Math. (Brno) 38, pp.201-208.

[37] Piri, H., Kumam, P., (2014), Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl., 2014:210, 11p.

[38] Rhoades, B., Singh, S.L., Kulshrestha, C., (1984), Coincidence theorems for some multivalued mappings, Internat. J. Math. Math. Sci., 7(3), pp.429-434.

[39] Samet, B., Vetro, C., (2011), Comments on the paper "Coincidence theorems for some multivalued mappings" by B. E. Rhoades, S. L. Singh, Chitra Kulshrestha, Fasc. Math., 47, pp.89-94.

[40] Sessa, S., (1982), On a weak commutativity condition in fixed point considerations, Publ. Inst. Math. (Beograd), 32(46), pp.149-153.

[41] Sgroi, M., Vetro, C., (2013), Multi-valued F-contractions and the solution of certain functional and integral equations, Filomat 27(7), pp.1259-1268.

[42] Shahzad, N., (2003), Coincidence points and R-subweakly commuting multivalued maps, Demonstratio Math., 36(2) pp.427-431.

[43] Singh, S.L., Hashim, A.M., (2005), New coincidence and fixed point theorems for strictly contractive hybrid maps, Aust. J. Math. Anal. Appl., 2(1), Art.12, 7 p.

[44] Singh, S.L., Mishra, S.N., (1997), Remarks on Jachymski's fixed point theorems for compatible maps, Indian J. Pure Appl. Math., 28(5), pp.611-615.

[45] Singh, S.L., Mishra, S.N., (2001), Coincidences and fixed points of nonself hybrid contractions, J. Math. Anal. Appl., 256(2), pp.486-497.

[46] Sintunavarat, W., Kumam, P., (2009), Coincidence and common fixed points for hybrid strict contractions without the weakly commuting condition, Appl. Math. Lett. 22, pp.1877-1881.

[47] Sintunavarat, W., Kumam, P., (2011), Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math., Article ID 637958, 14 p.

[48] Sintunavarat, W., Kumam, P., (2012), Common fixed point theorem for hybrid generalized multi-valued contraction mappings, Appl. Math. Lett., 25, pp. 52–57.

[49] Suzuki, T., (2005), Generalized Caristi's fixed point theorems by Bae and others, J. Math. Anal. Appl. 302(2), pp. 502–508.

[50] Vetro, F., (2016), F-contractions of Hardy-Rogers type and application to multistage decision processes, Nonlinear Anal. Model. Control, 21(4), pp.531-546.

[51] Wardowski, D., (2012), Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl., 2012:94, 6 pp.

[52] Zhao, J., Liang, Y., Liu, Y., Cho, Y. J., (2018), Split equilibrium, variational inequality and fixed point problems for multi-valued mappings in Hilbert spaces, Appl. Comput. Math., 17(3), pp.271-283.

Hemant Kumar Nashine - is an expert in the field of fixed point theory (nonlinear functional analysis) and its applications to differential and integral equations. Presently he is working as Professor in VIT Vellore India and visiting professor in University of Johannesburg South Africa.

Mohammad Imdad - is a Professor of Mathematics at Department of Mathematics, Aligarh Muslim University, Aligarh, India. His area of research interests includes general topology, functional analysis, fixed point theory, fuzzy set theory, operator theory and integral equations.

Md. Ahmadullah - is an Assistant Professor in the Department of Mathematics, Rammohan College (Affiliated to Calcutta University), Kolkata, West Bengal, India. Dr. Ahmadullah received his M.Sc., B.Ed. and Ph.D. degrees in mathematics from Aligarh Muslim University, Aligarh. His research area is operator theory especially fixed point theory.